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Weighted Values and the Core

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1 Introduction

The main result of this paper is that the set of all weighted Shapley values of a cooperative game contains the core of the game. That there is such a general relationship between core and values is somewhat surprising in light of the difference in concept behind these solutions. Indeed cooperative game theory tells us very little about the relations between core and values. Such relations are known to exist for convex games and for market games with a continuum of players. In such games the Shapley value is always in the core. Convex games have, in a sense, large cores, which 'explains' why they contain the Shapley value. In the case of market games with a continuum of players, it is the homogeneity of the games and the diagonal property of the Shapley value that guarantee this fact.

More relations can be found when we consider core-like and value-like solutions. In a recent paper Owen (1990) shows that for spatial voting games the Copeland winner outcome, which is a near core solution concept, is an analogue of the Shapley value. A result concerning a relation between the core and value-like solutions for general games was noted by Weber (1988) who showed that the set of all random order values of a game contains the core. Our result generalizes Weber's since weighted values constitute a subset (dimensionally, a very small one) of random order values.

Weighted Shapley values (weighted values for short) were defined by Shapley (1953a, b) alongside the standard Shapley value and were extensively discussed in the literature (e.g., Owen (1972), Kalai, and Samet (1987), and Hart and Mas-Colell (1989)). For these values weights are assigned to the players. The value is then determined in one of two equivalent ways. In the random order approach the weights are used to determine a probability distribution over orders of the players and the value is the expected contributions of the players according to this probability distribution. In the algebraic approach the value of a unanimity game is determined first, by allocating one unit among the players of the carrying coalition according to their

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relative weights. The value for general games is extended then by linearity. Weighted values were axiomatized by Shapley (1981), using the weights explicitly in the axioms, and by Kalai and Samet (1987) without doing so. Hart and Mas-Colell (1989) used the potential approach in order to provide new axiomatizations for the weighted values.

Our first results concern convex games. We characterize convex games by a property of the set of all weighted values of a game. We show that the weighted values of a given game are monotonic with respect to the weights if and only if the game is convex. By monotonicity of the weighted values for a given game we mean that when a player's weight is increased, while keeping the other players' weights unchanged, the player's value in the given game increases. The intuition behind this claim is as follows. When we examine the dependence on the weights of the probability distribution over orders we find that by increasing a player's weight we increase his chances to arrive 'late'. Convex games are precisely those games in which a player's contribution increases when he arrives 'late'. Therefore by increasing a player's weight we increase his expected contribution. Using the monotonicity property we prove that a game is convex if and only if its core coincides with the set of all its weighted values. This last result is used together with a fixed-point argument to prove the main result of the paper. The core of any game is a subset of the set of all its weighted values.

The difficulty in studying weighted values stems from the special structure of the family of weighted values as a set of linear operators from the space of games into R^N where N is the set of players. The association of each positively weighted value, namely one in which each player has a positive weight, with the corresponding weight vector is a homeomorphism between the positively weighted values and the relative interior of the unit simplex in R^N. But, unlike the latter, the set of positively weighted values is not a convex set. Moreover, this natural homeomorphism cannot be extended to one between the closure of the positively weighted values, which is the set of all weighted values, and the unit simplex. This is so because the limit of a sequence of positively weighted values depends on the rate of convergence to zero of different weights.

Each game v can be viewed naturally as a linear transformation from the space of linear solutions to R^N. It maps the set of all weighted values to the set of all payoff vectors assigned to v by all the weighted values. The non-trivial structure of the payoff set is inherited from the non-trivial structure of the set of weighted values which is mapped on it linearly by v. Since the set of all weighted values is not convex we cannot expect it to be mapped onto a convex set by a linear mapping, and indeed in general it is not.

The structure of the set of weighted values is best revealed when it is mapped to R^N by a strictly convex game. In this case we show that the set of weighted values is mapped homeomorphically onto the core of the game. Moreover, the structure of the core, which was studied by Shapley (1971) is reflected in a natural way in the structure of the set of weighted values.

The set of weighted values can be easily shown to be homeomorphic to the set of conditional systems that was discussed in the literature of non-cooperative game theory (e.g., see Myerson (1986) and Mclennan (1989a, b)). The main theorem of Mclennan (1989b) states that this set is homeomorphic to a ball. As the core of a strictly convex game is homeomorphic to a ball, this paper provides an independent proof to Mclennan's theorem.

The lack of convexity of the set of weighted values distinguishes it from the much simpler set of random order values. Since the latter is a convex set of values it follows that the set of all random order values of a game is also convex. The proof that this set contains the core requires standard techniques of convex analysis. By contrast, the case of weighted values seems to require heavier tools. The proof of our main result relies on a fixed-point theorem.

2 Preliminaries

Let N be a finite set with $n \geq 1$ elements which we call *players*. The set of all nonempty subsets of N will be denoted by $\mathscr{M}$. For $S \subseteq N$ we denote $N \backslash S$ by S^c and for $i \in N$ we write $S \cup i$ for $S \cup \{i\}$ and $S \backslash i$ for $S \backslash \{i\}$. For $S \in \mathscr{M}$ we write x^S, y^S, etc., for elements in R^S. If $S \subset T$ then x^T_S is the projection of x^T on R^S.

For $x, y \in R^N$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x_i > y_i$ for all $i \in N$.

For $x \in R^N$ and $S \in \mathscr{M}$ we write $x(S)$ for $\sum_{i \in S} x_i$; for $S = \emptyset$, $x(S) = 0$. For a finite set X, $\Delta(X)$ is the unit simplex in R^X. We denote by $Int(\Delta(X))$ the relative interior of $\Delta(X)$.

We denote by $\mathscr{R}$ the set of all $n!$ (complete) orders on N. For $i, j \in N$ and for $r \in \mathscr{R}$ we shall write $i <_r j$, if i comes before j in the order r. The set of all players preceding player i in r is denoted by Q^i_r. A *game* on N is a function $v: 2^N \rightarrow R$ with $v(\emptyset) = 0$. The space of all games on N will be denoted by G. A game v is *convex* if for every $T \subset S$ and for every $i \notin S$, $v(T \cup i) - v(T) \leq v(S \cup i) - v(S)$. v is *strictly convex* if the last inequalities are always strict. For every $S \in \mathscr{M}$ the *unanimity game* u_S is defined as follows: $u_S(T) = 1$ if $S \subseteq T$, and $u_S(T) = 0$ otherwise. The *core* $C(v)$ of v is the set of all $x \in R^N$ for which $x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subseteq N$.

The *contribution vector* $z^v(r) \in R^N$ for the game v and the order r is defined by $z^v(r)_i = v(Q^i_r \cup i) - v(Q^i_r)$ for each $i \in N$. Let P be a probability measure over all $n!$ orders of N. i.e., $P \in \Delta(\mathscr{R})$. The *P-random order value* $\psi(P): G \rightarrow R^N$ is defined by $\psi^v(P) = E_P(z^v(.))$ for each $v \in G$, where E_P is the expectation with respect to P. That is,

$$\psi^v_i(P) = \sum_{r \in \mathscr{R}} (v(Q^i_r \cup i) - v(Q^i_r))P(r) \quad \text{for all } v \in G \text{ and for all } i \in N. \tag{2.1}$$

The structure of the core of a convex game is described in Shapley (1971) (see also Ichiishi (1983)) as follows.

Let $\sum$ be the set of all ordered partitions of N. Elements of $\sum$ are of the form $\sigma = (S_1, \ldots, S_k)$, where $k \geq 1$, $\cup^k_{h=1} S_h = N$, and for all $i \neq j \in N$, $S_i \cap S_j = \emptyset$ and $S_i \neq \emptyset$.

For each $\sigma = (S_1, \ldots, S_k)$ in $\sum$ let F^v_σ be the set of all $x \in C(v)$ such that $x(\cup^h_{j=1} S_j) = v(\cup^h_{j=1} S_j)$ for all $1 \leq h \leq k$. When v is convex, F^v_σ is a nonempty face of $C(v)$ of dimension $n-k$ at most. In particular for σ with $k=n$, F_σ^v consists of one point. This point is a contribution vector, and $C(v)$ is the convex hull of all contribution vectors. When v is strictly convex then F_σ^v is of dimension $n-k$ and all faces F_σ^v are distinct. Finally, the core of v contains all the contribution vectors if v is convex; if in addition all the contribution vectors are distinct then the game is strictly convex.

3 Positively Weighted Values

For a vector $\omega \in R_{++}^N$ (i.e., $\omega \gg 0$), the *positively weighted value* $\varphi(\omega): G \to R^N$ is defined in Shapley (1953a) as the unique linear operator satisfying for each unanimity game u_S:

$$\varphi_i^{u_S}(\omega) = \begin{cases} \omega_i/\omega(S), & \text{for } i \in S \\ 0, & \text{for } i \notin S. \end{cases} \tag{3.1}$$

Since $\varphi^{(\cdot)}(\omega)$ is linear, (3.1) defines $\varphi^v(\omega)$ for all $v \in G$.

If $\omega_i = \frac{1}{n}$ for all $i \in N$ then $\varphi(\omega)$ is the Shapley value.

Owen (1972) has shown that the positively weighted Shapley values are random order values. For a weight vector ω we define P_ω in $\Delta(\mathscr{R})$ as follows. Let X_i for $i \in N$ be independent random variables distributed over [0, 1] such that for every $0 \le t \le 1$ and $i \in N$,

$$Prob(X_i \le t) = t^{\omega_i}.$$

For $r \in \mathscr{R}$ define:

$$P_\omega(r) = P(X_i < X_j \quad \text{for all } i <_r j,\ i, j \in N). \tag{3.2}$$

Then for every $v \in G$, $\varphi^v(\omega) = \psi^v(P_\omega)$. Note that both $\varphi^v(.)$ and $P_{(.)}$ are positively homogeneous of degree one. Therefore we can restrict our attention to vectors ω in $Int(\Delta(N))$, that will be called *weight vectors*.

4 Weighted Values

We generalize now the notion of a weight vector to enable some players to have zero weight. The values corresponding to the generalized weights will be called *weighted values*. These values were defined by Shapley (1953b) and axiomatized by Kalai and Samet (1987). The following consideration will lead us to this generalization.

When zero-weight players are allowed we can not use directly (3.1) to define a value, since for S which contains only zero-weight players (3.1) is not defined. We need therefore to assign secondary weights to the zero-weight players, that will be used for coalitions which contain only zero-weight players. These new secondary weights may themselves assign zeros for some of the players and we have to assign also weights to these doubly zero-weighted players, and so on. We are naturally lead to the following definition.

A *generalized weight vector* is a $2k$-tuple, $1 \leq k \leq n$, $(S_1, \ldots, S_k, w^{S_1}, \ldots, w^{S_k})$ such that $(S_1, \ldots, S_k) \in \sum$ and $w^{S_h} \in Int(\Delta(S_h))$ for $h = 1, \ldots, k$. The interpretation of the generalized weight vectors is as follows. The players in S_k are the non-zero weight players with weights given by w^{S_k}, while the rest of the players are zero-weight players. Among the zero-weight players the 'heaviest' are the members of S_{k-1} with weights $w^{S_{k-1}}$. All players in $\cup_{h \leq k-2} S_h$ are zero-weight players relative to players in S_{k-1}, etc. Note that every weight vector ω can be naturally identified with the generalized weight vector (N, ω).

Given the generalized weight vector $(S_1, \ldots, S_k, w^{S_1}, \ldots, w^{S_k})$ we can find the relative weights of player in each coalition S. For a given S let $m = \max\{h \mid S_h \cap S \neq \emptyset\}$. Define now w^S by: $w_i^S = w_i^{S_m} / w^{S_m}(S \cap S_m)$ for $i \in S \cap S_m$ and $w_i^S = 0$ for $i \in S \backslash S_m$. Thus $S \cap S_m$ consists of all the 'heaviest' players in S. The relative weights of these players are determined by their weights in S_m. The rest of the players have zero weight in S.

Denote now $w = (w^S)_{S \in \mathscr{M}}$. w is a vector in $\prod_{S \in \mathscr{M}} \Delta(S)$ and it is easy to verify that it satisfies:

$$w_i^S = w_i^T / w^T(S) \tag{4.1}$$

for each $i \in S \subseteq T$ such that $w^T(S) > 0$.

A vector $w \in \prod_{S \in \mathscr{M}} \Delta(S)$ which satifies (4.1) is called a *weight system*. The set of all weight systems is denoted by $\mathscr{W}$. We saw that each generalized weight vector corresponds to a weight system. It is easy to see that this correspondence is one to one. We show now that it is also onto $\mathscr{W}$.

Let $w \in \mathscr{W}$ and define $\sigma(w) \in \sum$ as follows. Let $T_1 = \{i \in N : w_i^N > 0\}$, and for $h \geq 2$ we define T_h to be the set

$$\left\{ i \in (\cup_{j=1}^{h-1} T_j)^c : w_i^{(\cup_{j=1}^{h-1} T_j)^c} > 0 \right\}$$

when this set is not empty.

Let T_k be the last nonempty set so defined, then clearly $(T_1, \ldots, T_k)$ is an ordered partition. Now for each $1 \leq h \leq k$, let $S_h = T_{k-h+1}$ and $\sigma(w) = (S_1, \ldots, S_k)$. It is easy to see now that w is the weight system that corresponds to the generalized weight vector $(S_1, \ldots, S_k, w^{S_1}, \ldots, w^{S_k})$.

For a given $\sigma \in \sum$ we denote by $\mathscr{W}_\sigma$ the set of all weight systems w for which $\sigma(w) = \sigma$. Note that for $w \in \mathscr{W}$, $w \in \mathscr{W}_{(N)}$ iff $w^N \gg 0$, and that $\mathscr{W}_{(N)}$ is dense in $\mathscr{W}$.

For a given $w \in \mathscr{W}$ we define now the *weighted value* $\varphi(w)$ as the linear function $\varphi(w)$: $G \to R^N$ which is defined for each unanimity game u_S by:

$$\varphi_i^{u_S}(w) = w_i^S \quad \text{for } i \in S \text{ and } \varphi_i^{u_S}(w) = 0 \text{ otherwise.}$$

Note that for $w \in \mathscr{W}_{(N)}$ the weighted value $\varphi(w)$ coincides with the positively weighted value $\varphi(w^N)$.

Weighted values are also random order values. The probability distribution P_w in $\Delta(\mathscr{R})$ which defines $\varphi(w)$ is described as follows.

We say that an order r of N is *consistent* with $\sigma = (S_1, S_2, \ldots, S_k)$ in $\sum$ if for each $1 \leq h \leq k-1$ each player in S_h precedes each player in S_{h+1}. For each $w \in \mathscr{W}$ we define a probability measure P_w over all orders of N, the support of which is the set of all orders which are consistent with $\sigma(w)$.

Now let $(S_1, \ldots, S_k, w^{S_1}, \ldots, w^{S_k})$ be the generalized weight vector which corresponds to w. Since $w^{S_h} \in Int(\Delta(S_h))$ for all $1 \leq h \leq k$, we can define for each such h a probability distribution $P_{w^{S_h}}$ on the orders of S_h in the same way P_{w^N} was defined in (3.2). We define now

$$P_w(r) = \prod_{h=1}^{k} P_{w^{S_h}}(r_h),$$

where r_h is the order on S_h induced by r. P_w is the probability distribution for which $\varphi^v(w) = \psi^v(P_w)$ for each game v.

Notice that in all the orders which are consistent with $\sigma(w)$, the non-zero players (those in S_k) are preceded by all other players, all the players in S_{k-1} are preceded by the players in $\cup_{h \leq k-2} S_h$ etc.

5 The Main Results

The main theorem is the following:

Theorem A. For every game v, each element in the core of v is the weighted value of v for some weight system. That is, $C(v) \subseteq \varphi^v(\mathscr{W})$.

The *anti-core* $AC(v)$ of the game v is defined to be the set of all $x \in R^N$ for which $x(N) = v(N)$, and $x(S) \leq v(S)$ for all $S \subseteq N$. The anti-core is a natural solution concept for games that model cost allocation problems. Note that $AC(v) = -C(-v)$ for all $v \in G$. Therefore, Theorem A and the linearity property of the weighted values imply:

Theorem B. For every game v, each element in the anti-core of v is the weighted value of v for some weight system. That is, $AC(v) \subseteq \varphi^v(\mathscr{W})$.

More specific results are obtained for convex games.

Theorem C. A game v is convex iff $C(v)=\varphi^v(\mathscr{W})$. Moreover, v is strictly convex iff φ^v is a homeomorphism between $\mathscr{W}$ and $C(v)$; In this case for each $\sigma\in\sum$, φ^v maps homeomorphically $\mathscr{W}_\sigma$ onto the relative interior of the face F^v_σ of $C(v)$.

A corollary of Theorem C is:

Corollary C. The set $\mathscr{W}$ of all weight systems on N is homeomorphic to a $n-1$ dimensional ball, where n is the cardinality of N.

Weber (1988) proved that for every game v

$$C(v)\subseteq\psi^v(\Delta(\mathscr{R})), \tag{5.1}$$

where $\psi^v(\Delta(\mathscr{R}))=\{\psi^v(P):P\in\Delta(\mathscr{R})\}$ is the set of all random order values of v. Since weighted values are random order values $\varphi^v(\mathscr{W})\subseteq\psi^v(\Delta(\mathscr{R}))$ and therefore Theorem A implies Weber's result. Moreover, the set of all weighted values has the dimension of $\mathscr{W}$ which according to Corollary C is $n-1$. The set of all random order values can be shown to have the dimension $2^{n-1}(n-2)+1$. Thus Theorem A shows that a much "thiner" set of values is required in order to cover the core of each game. Note also that $\varphi^v(\mathscr{W})$ is not a convex set in general and therefore it is strictly contained in the set $\psi^v(\Delta(\mathscr{R}))$.

The set $\mathscr{W}$ has been discussed in the literature of non-cooperative game theory under the title 'the space of conditional systems' (e.g., see Myerson (1986) and Mclennan (1989a, b)). Condition (4.1) can be phrased as saying that for each S, w^S is the conditional probability on S derived from the probability distributions on supersets of S whenever such derivation is possible. The result of Mclennan (1989b) states that $\mathscr{W}$ is homeomorphic to a ball and it is proved using algebraic topology techniques. Corollary C is an independent proof of this result. It shows that $\mathscr{W}$ is homeomorphic to the core of any strictly convex game. Convex games can be also characterized by another property of the weighted values.

For every $i\in N$ and for every $w\neq u$ in $\mathscr{W}$ we write $w>_i u$ if $w^S_i\geq u^S_i$ for every S which contains i and $w^S=u^S$ for all $S\subseteq N\backslash i$.

We say that φ^v is *increasing* if for each i, for each ordered partition σ, and for each w, $u\in\mathscr{W}_\sigma$ such that $w>_i u$,

$$\varphi^v_i(w)\geq\varphi^v_i(u). \tag{5.2}$$

φ^v is *strictly increasing* if the inequalities in (5.2) are strict.

Theorem D. The game v is (strictly) convex iff φ^v is (strictly) increasing.

6 Proofs

We will need the following lemma:

Lemma 1. Let v be a convex game and let $w >_i u$, where $w, u \in \mathscr{W}_{(N)}$. Then

$$\varphi_j^v(w) \leq \varphi_j^v(u) \quad \textit{for all } j \neq i. \tag{6.1}$$

Moreover, if v is strictly convex then the inequalities in (6.1) are strict.

Proof: It can be easily verified that it suffices to prove that for a convex game v,

$$\frac{\partial \varphi_j^v(\omega)}{\partial \omega_i} \geq 0 \quad \text{for all } \omega \in R_{++}^N, \tag{6.2}$$

and that for a strictly convex game v the inequalities in (6.2) are strict.

Indeed, let f^v be the multilinear extension of v as defined by Owen (1972). Then

$$\varphi_j^v(\omega) = \int_0^1 \frac{\partial f^v}{\partial x_j}(t^{\omega_1}, t^{\omega_2}, \dots, t^{\omega_n})\,\omega_j t^{\omega_j - 1}\, dt.$$

Therefore,

$$\frac{\partial \varphi_j^v(\omega)}{\partial \omega_i} = \int_0^1 \frac{\partial^2 f^v}{\partial x_j \partial x_i}(t^{\omega_1}, t^{\omega_2}, \dots, t^{\omega_n})\,\omega_j t^{\omega_j + \omega_i - 1} \ln(t) dt. \tag{6.3}$$

It is well-known that v is convex iff for all $i \neq j$

$$\frac{\partial^2 f^v}{\partial x_j \partial x_i}(x) \geq 0 \quad \text{for all } x \in [0, 1]^N,$$

and that v is strictly convex iff for all $i \neq j$

$$\frac{\partial^2 f^v}{\partial x_j \partial x_i}(x) > 0 \quad \text{for all } x \in [0, 1]^N.$$

Hence the result follows from (6.3). ■

Proof of Theorem D: Lemma 1 and the efficiency property of φ (i.e., $\varphi^v(w)(N) = v(N)$) imply that for a convex game v, $\varphi_i^v(w) \geq \varphi_i^v(u)$ whenever $w, u \in \mathscr{W}_{(N)}$ and $w >_i u$, and that for a strictly convex game v the last inequalities are strict.

Let then $\sigma=(S_1, S_2, \dots, S_k)$ be an ordered partition different from (N), w, $u\in\mathscr{W}_\sigma$ with $w>_i u$, and let v be a convex game. Suppose $i\in S_h$. Observe that S_h contains at least two players because $w>_i u$. Define a game v_h on the grand coalition S_h by

$$v_h(T)=v(T\cup_{k<h}S_k)-v(\cup_{k<h}S_k) \quad \text{for all } T\subseteq S_h. \tag{6.4}$$

It can be easily verified that for each $\tau\in\mathscr{W}_\sigma$

$$\varphi^v(\tau)_{S_h}=\varphi^{v_h}(\tau_{S_h}), \tag{6.5}$$

where $\varphi^v(\tau)_{S_h}=(\varphi^v_k(\tau))_{k\in S_h}$ and $\tau_{S_h}=(\tau^S)_{S\subseteq S_h}$.

Note that v_h is (strictly) convex whenever v is (strictly) convex, and since $w^{S_h}\gg 0$ and $u^{S_h}\gg 0$ the result follows from what we showed for weight systems in $\mathscr{W}_{(N)}$.

We have shown that (strict) convexity of v implies (strict) monotonicity of φ^v. Conversely, assume φ^v is increasing and that v is not convex. Then there exist $i\neq j$ and $S\subseteq\{i,j\}^c$ such that

$$v(S\cup i)-v(S)>v(S\cup j\cup i)-v(S\cup j). \tag{6.6}$$

Consider w such that $\sigma(w)=(S, \{i,j\}, (S\cup i\cup j)^c)$. Then by (6.5) and the formula for computing the positively weighted value for 2-person games,

$$\varphi^v_i(w)=w_i^{\{i,j\}}[v(S\cup j\cup i)-v(S\cup j)]+w_j^{\{i,j\}}[v(S\cup i)-v(S)]. \tag{6.7}$$

Consider u defined by $u^T=w^T$ for each $T\neq\{i,j\}$, $u_i^{\{i,j\}}=w_i^{\{i,j\}}-\varepsilon$ and $u_j^{\{i,j\}}=w_j^{\{i,j\}}+\varepsilon$ for small enough $\varepsilon>0$. Clearly $w>_i u$ and $\sigma(u)=\sigma(w)$. Thus we can write for u an expression similar to (6.7). From (6.6) and (6.7)

$$\varphi^v_i(w)<\varphi^v_i(u). \tag{6.8}$$

This contradicts the monotonicity of φ^v.

If v is assumed not to be a strictly convex game then we can guarantee only weak inequality in (6.6) and therefore weak inequality in (6.8). This is, however, sufficient to contradict strict monotonicity of φ^v. This complete the proof of Theorem D. ■

We prove Theorem C through Lemmas 2–4.

Lemma 2. Let v be a convex game. Then for each $\sigma\in\sum$, $\varphi^v(\mathscr{W}_\sigma)\subseteq F^v_\sigma$. Moreover, if v is strictly convex then $\varphi^v(\mathscr{W}_\sigma)\subseteq Int(F^v_\sigma)$ – the relative interior of F^v_σ.

Proof: Let $\sigma=(S_1, \dots, S_k)$ and let $w\in\mathscr{W}_\sigma$. Observe that for every game v,

$$\varphi^v(w) = \sum_{r\in\mathscr{R}_\sigma} z^v(r)P_w(r), \tag{6.9}$$

where $\mathscr{R}_\sigma$ is the set of all orders which are consistent with σ.

Suppose v is a convex game and let $r \in \mathscr{R}$ then by Shapley (1971), $z^v(r)(S) \geq v(S)$ for all $S \in \mathscr{M}$ and $z^v(S) = v(S)$ for every S which is an initial segment with respect to r (i.e., $S = Q_r^i$ for some $i \in N$). In particular, for $r \in \mathscr{R}_\sigma$,

$$z^v(r)(S^j) = v(S^j) \quad \text{for all } 1 \leq j \leq k,$$

where $S^j = \cup_{h \leq j} S_h$ for every $1 \leq j \leq k$. Hence $\varphi^v(\omega) \in F_\sigma^v$ by (6.9).

By Shapley (1971), for a strictly convex game v, $z^v(r)(S) > v(S)$ for every coalition S which is not an initial segment with respect to r. As $P_w(r) > 0$ for all $r \in \mathscr{R}_\sigma$, (6.9) implies that $\varphi^v(w)(S) > v(S)$ for all $S \notin \{S^1, S^2, \ldots, S^k\}$. Therefore $\varphi^v(w) \in Int(F_\sigma^v)$. ■

Lemma 3. Let v be a strictly convex game. Then φ^v is one-to-one.

Proof: Since relative interiors of different faces of $C(v)$ are disjoint it suffices by Lemma 2 to prove that φ^v is 1–1 on each $\mathscr{W}_\sigma$. Let then $\sigma = (S_1, \ldots, S_k)$ and let $w \neq u \in \mathscr{W}_\sigma$. Then for some $1 \leq h \leq k$, $w^{S_h} \neq u^{S_h}$. In order to show that $\varphi^v(w) \neq \varphi^v(u)$ it suffices by (6.5) to show that $\varphi^{v_h}(w_{S_h}) \neq \varphi^{v_h}(u_{S_h})$, where v_h is defined in (6.4). Since v_h is strictly convex, $w^{S_h} \gg 0$, and $u^{S_h} \gg 0$, it suffices to prove that weighted values of strictly convex games are one-to-one on $\mathscr{W}_{(N)}$.

We now show that if $a, b \in R_{++}^N$, $a < b$, and for some j, $a_j = b_j$, then $\varphi_j^v(a) > \varphi_j^v(b)$. Suppose firstly that for some $i \neq j$, $a_i < b_i$ and $a_k = b_k$ for each $k \neq i$. Then by Lemma 1, in this case, $\varphi_j^v(a) > \varphi_j^v(b)$. In the general case, one can find a sequence $a^1, a^2, \ldots, a^l$ such that $a^1 = a$, $a^l = b$ and a^{k+1} is obtained from a^k by increasing one coordinate only.

Now for $w \neq u$ in $\mathscr{W}_{(N)}$ define $\lambda_0 = \min\{\lambda > 0 : \lambda w^N \geq u^N\}$ and let $b = \lambda_0 w^N$. Then $b > u^N$ and there exists j such that $b_j = u_j^N$. Therefore,

$$\varphi^v(w) = \varphi^v(b) \neq \varphi^v(u^N) = \varphi^v(u).$$ ■

Lemma 4. Let v be a strictly convex game. Then $\varphi^v(\mathscr{W}) = C(v)$.

Proof: In what follows we denote the relative interior of a convex set X by $Int(X)$. Recall that $C(v) = F_{(N)}^v$. Therefore, by Lemma 2, φ^v maps $\mathscr{W}_{(N)}$ into the relative interior of $C(v)$. Since the closure of $\mathscr{W}_{(N)}$ is $\mathscr{W}$ and φ^v is continuous, it suffices to prove that $\varphi^v(\mathscr{W}_{(N)}) = Int(C(v))$. To see this note that $\mathscr{W}_{(N)}$ is homeomorphic to $Int\Delta(N)$ and therefore, both $\mathscr{W}_{(N)}$ and $Int(C(v))$ are homeomorphic to R^{n-1}. Since φ^v is continuous and 1–1, we deduce from the Invariance of Domain Theorem (e.g., see Istratescu (1981)) that $\varphi^v(\mathscr{W}_{(N)})$ is open in $Int(C(v))$. Let $B = Int(C(v)) \setminus \varphi^v(\mathscr{W}_{(N)})$. We now show that B is open in $Int(C(v))$ and therefore must be empty since $Int(C(v))$ is connected. Indeed, if B is not open then there exists $z \in B$ and a sequence (z^m), $m \geq 1$ in $\varphi^v(\mathscr{W}_{(N)})$ such that $z^m \to z$. For each m there exists w^m in $\mathscr{W}_{(N)}$ such that $\varphi^v(w^m) = z^m$. Since $\mathscr{W}$ is compact we may assume without loss of generality that $w^m \to w \in \mathscr{W}$. Since φ^v is continuous $\varphi^v(w) = z$ and hence $w \notin \mathscr{W}_{(N)}$ and $w \in \mathscr{W}_\sigma$ for some $\sigma \neq (N)$. But then, by Lemma 2, $z \in Int(F_\sigma^v)$ which is a contradiction since $Int(F_\sigma^v) \cap Int(F_{(N)}^v) = \emptyset$. ■

Proof of Theorem C: By Lemmas 3 and 4, for a strictly convex v, φ^v is a 1–1 continuous map from $\mathscr{W}$ onto $C(v)$ and since $\mathscr{W}$ is compact it is a homeomorphism. By Lemmas 2, 3, and 4, $\varphi^v: \mathscr{W}_\sigma \to F_\sigma^v$ is a 1–1 continuous map onto $Int(F_\sigma^v)$ and therefore it is a homeomorphism.

Now let v be a convex game. There exists a sequence $(v^m)_{m=1}^\infty$ of strictly convex games on N such that $v^m(S) \to v(S)$ for each $S \subseteq N$. For each m, $\varphi^{v^m}(\mathscr{W}) = Co\{z^{v^m}(r)\}$, were Co stands for "convex hull" and r ranges over all orders of N. Since $z^{v^m}(r) \to z^v(r)$ and $\varphi^v(w)$ is continuous in both w and v, $\varphi^v(\mathscr{W}) = Co\{z^v(r)\} = C(v)$.

Conversely, suppose $\varphi^v(\mathscr{W}) = C(v)$. For each order r of N let $\sigma_r = (\{i_1\}, \{i_2\}, \ldots, \{i_n\})$, where $i_1, i_2, \ldots, i_n$ is the order of the players according to r. $\mathscr{W}_{\sigma_r}$ consists of a single weight system w_r and $\varphi^v(w_r) = z^v(r)$. Thus $z^v(r) \in C(v)$ for each order r and therefore v is convex. Moreover, if φ^v is a homeomorphism then all $n!$ vectors $z^v(r)$ are distinct and therefore v is strictly convex. ■

Proof of Theorem A: Let v be a game with a nonempty core and let $x \in C(v)$. We have to show that there exists $w \in \mathscr{W}$ for which $\varphi^v(w) = x$.

Let u be a fixed strictly convex game such that $v+u$ is also a strictly convex game. By Theorem C, φ^{v+u} has a continuous inverse function that we denote by g. That is, $g: C(v+u) \to \mathscr{W}$.

For each $w \in \mathscr{W}$, $\varphi^u(w) \in C(u)$ and since $x \in C(v)$, $(x + \varphi^u(w)) \in C(v+u)$.

Define $f: \mathscr{W} \to \mathscr{W}$ by

$$f(w) = g(x + \varphi^u(w)) \quad \text{for all } w \in \mathscr{W}.$$

Since f is continuous, it follows from Corollary C that it has a fixed point in $\mathscr{W}$, say w_0. Hence, $\varphi^{v+u}(w_0) = x + \varphi^u(w_0)$, and from the linearity property of $\varphi^{(\cdot)}(w_0)$, $\varphi^v(w_0) = x$. ■

7 Dual Weighted Values

Weight systems can be used to define a different family of values which we call *dual weighted values*. For $w \in \mathscr{W}$ the dual weighted value $\varphi^*(w)$ is defined by $(\varphi^*)^v(w) = \varphi^{v^*}(w)$ where v^* is the dual game of v which satisfies for each S: $v^*(S) = v(N) - v(N \backslash S)$.

Dual positively weighted values were used by Shapley (1981) for cost allocation problems, where he also provided an axiomatization of $\varphi^*(w)$ for a fixed positive w. An axiomatization of the whole family of the dual weighted values was given by Kalai and Samet (1987). Like weighted values, Dual weighted values are also random order values. The probability distribution P_w^* which determines $\varphi^*(w)$ assigns to each order r the probability $P_w(r^*)$ where P_w is the probability distribution (described in Section 3) which defines $\varphi(w)$ and r^* is the order r reversed.

The families of the weighted values and of the dual weighted values intersect. Thus for example, the random order values that are defined by probability distributions on $\mathscr{R}$ that are concentrated on a single order in $\mathscr{R}$ belong to both families. However, Kalai and Samet (1987) proved that for $n \geq 3$, the Shapley value is the unique element in the intersection of the positively weighted values and the dual positively weighted values.

Note that for every game v, $C(v) = AC(v^*)$. Therefore we get the analogues of theorems A and B as follows:

Theorem A. For every game v, each element in the core of v is the dual weighted value of v for some weight system. That is,* $C(v) \subseteq (\varphi^*)^v(\mathscr{W})$.

Theorem B. For every game v, each element in the anti-core of v is the dual weighted value of v for some weight system. That is,* $AC(v) \subseteq (\varphi^*)^v(\mathscr{W})$.

Observe further that a game v is (strictly) convex iff $-v^*$ is (strictly) convex, and that for each game v, $\varphi^{-v}(\mathscr{W}) = -\varphi^v(\mathscr{W})$. Hence, Theorem C implies the next Theorem:

Theorem C. A game v is convex iff* $C(v) = (\varphi^*)^v(\mathscr{W})$. *Moreover, v is strictly convex iff* $(\varphi^*)^v$ *is a homeomorphisim between* $\mathscr{W}$ *and* $C(v)$. *In this case for each* $\sigma \in \sum$, $(\varphi^*)^v$ *maps homeomorphically* $\mathscr{W}_\sigma$ *onto the relative interior of the face* $F^v_{\sigma^*}$ *of* $C(v)$, *where* σ^* *is ordered partition* σ *reversed.*

We say that $(\varphi^*)^v$ is *decreasing* if for each i, for each ordered partition σ, and for each w, $u \in \mathscr{W}_\sigma$ such that $w >_i u$,

$$(\varphi^*)^v_i(w) \leq (\varphi^*)^v_i(u). \tag{7.1}$$

$(\varphi^*)^v$ is *strictly decreasing* if the inequalities in (7.1) are strict. Thus we have:

Theorem D. The game v is (strictly) convex iff* $(\varphi^*)^v$ *is (strictly) decreasing.*

Theorems C and C* imply that for convex games v,

$$\varphi^v(\mathscr{W}) = (\varphi^*)^v(\mathscr{W}).$$

We conjecture that the above equality holds for every game v.

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